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The r-Rank of the groups of exceptional Lie type

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ABSTRACT

In this note, we prove the following result, settling a question raised at the end of [Borel & Serre, 1953], cf. [Borel, 1983 pp. 228 and 708]. A related result for Lie groups of type E_8 was recently proved by J.F. Adams.

THEOREM. *Let r be a prime and G a simple algebraic group of exceptional type over an algebraically closed field of characteristic* $\neq r$. *Let E be an elementary abelian r-subgroup of G of maximal rank. Then* $\text{rank}(E) = \text{Lie rank}(G)$ *with the exception of* $r=2$ *and* $G=G_2$, F_4, *the adjoint* E_7, *and* E_8, *in which cases* $\text{rank}(E) = \text{Lie rank}(G)+1$. *Moreover, E is unique up to conjugacy.*

1. THE PRIME 2

In this section we prove the following

THEOREM. *Let G be an algebraic group of type* G_2, F_4, E_6, *adjoint* E_7, *simply connected* E_7, *or* E_8 *over an algebraically closed field of characteristic* $\neq 2$, *and let E be an elementary abelian* 2*-group in G of maximal order. Then* $|E| = 2^3, 2^5, 2^6, 2^8, 2^7, 2^9$ *in the respective cases. Moreover, in each case any two such elementary abelian subgroups are conjugate.*

PROOF. By a theorem of [Springer & Steinberg, 1970], due to [Borel & Serre, 1953] in the Lie group case, E is a subgroup of $N_G(T)$ for some maximal torus T of G. In particular, $|E| \leq 2^l \cdot |W|$, where l is the Lie rank of G and $W = N_G(T)/T$, so E is finite. We shall deal with each case separately, although the arguments are similar. The idea is to produce a certain subgroup containing the preimage in N of a Sylow 2-subgroup of W.

G_2. Let J_1, J_2 be commuting (nonconjugate) fundamental SL_2's. We may take $T \leq D = J_1 J_2$. Moreover $N_D(T)/T$ contains a Sylow 2-subgroup of $W = N_G(T)/T$, so we may assume $E \leq D$. Let $Z(D) = \langle e \rangle$. Maximality of E then implies $E = \langle e, x_1 x_2, y_1 y_2 \rangle$, where $x_1, y_1 \in J_1$, $x_2, y_2 \in J_2$, $x_1^2 = x_2^2 = y_1^2 = y_2^2 = [x_1, y_1] = [x_2, y_2] = e$. It is clear that any two such groups are conjugate in D.

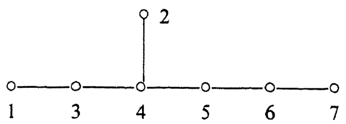
F_4. There is an involution in F_4 with centralizer D, the simply connected group of type B_4. We may take $T \leq D$ and check that $N_D(T)/T$ contains a Sylow 2-subgroup of $N_G(T)/T$. Hence, we may take $E \leq D$. An involution in SO_9 lifts to an involution in D if and only if the eigenspace for eigenvalue -1 has dimension a multiple of 4. A direct check then shows that D has 2-rank 5 and all elementary abelian subgroups of D of order 2^5 are conjugate.

E_6. Set $V = \Omega_2(T) = \{t \in T \mid t^2 = 1\}$. Then V is the natural module for $O^-(6,2) \cong W$. By § 8 of [Aschbacher & Seitz, 1976] W has 4 classes of involutions, represented by a_2, c_2, b_1, and b_3. Here the subscript is the dimension of the commutator space of the involution. The involutions in $\Omega^-(6,2)$ are conjugates of a_2 and c_2. Long root subgroups of W are generated by conjugates of a_2, the commutator space $[V, a_2]$ is totally singular, and $C_V(a_2) = [V, a_2]^\perp$. Finally, by (19.9)(ii) of [Aschbacher & Seitz, 1976], applied to $O^-(6,2) \cong U_4(2) \cdot 2$, we have $C_W(b_3) \leq C_W(b_1)$ for suitable choice of b_1.

To prove the theorem for E_6 we may and shall assume that E is not contained in a maximal torus of G, $|E| \geq 2^6$, and $E \leq N_G(T)$. Set $\bar{E} = ET/T$, and for $x \in E$, write $\bar{x} = xT$. If $\bar{E}$ centralizes b_1, then there is a fundamental SL_2 normalized by T, containing a preimage in $N(T)$ of b_1, and such that $E \leq SL_2 \circ SL_6$. A direct check then shows that E is necessarily contained in a maximal torus, a contradiction. Hence $\bar{E}$ does not centralize a b_1 involution. In particular, $\bar{E} \leq \Omega^-(6,2)$. Moreover, if $C_V(\bar{E})$ contains a nonsingular vector v, then $\bar{E}$ centralizes the unique involution b_1 of W satisfying $[V, b_1] = \langle v \rangle$, a contradiction. Therefore, $C_V(\bar{E})$ is totally singular, $|E \cap T| \leq 4$, and $|\bar{E}| \geq 2^4$.

Let $\bar{E} \leq P$ be the stabilizer in $O^-(6,2)$ of a singular 1-space of V. Then $P = O_2(P)L$, where $L \cong \Omega^-(4,2)$ and $O_2(P)$ is the natural module for L. Since $|\bar{E}| \geq 2^4$, an easy argument shows that $\bar{E} = O_2(P)$ and so $|\bar{E}| = 2^4$. Hence, $\bar{E}$ contains distinct a_2 involutions $\bar{x}, \bar{y}$. Then $C_V(\bar{x})$ and $C_V(\bar{y})$ have distinct radicals, so the singular points of $C_V(\bar{x}) \cap C_V(\bar{y})$ span a subspace of dimension ≤ 1. Consequently, $|E| \leq |\bar{E}| \cdot |E \cap T| \leq 2^4 \cdot 2 < 2^6$. This contradiction finishes the proof of the E_6 case.

E_7. Fix a maximal torus T and corresponding system of root groups. Let Σ denote a maximal set of pairwise commuting fundamental SL_2's from this system. If we label the diagram as follows



then we can take $\Sigma = \{J_1, \ldots, J_7\}$, where $J_i = \langle U_{\pm\beta_i} \rangle$ and the β_i are as follows:

$$\beta_1 = 2234321, \quad \beta_2 = 0112221, \quad \beta_3 = 0000001, \quad \beta_4 = 0112100,$$

$$\beta_5 = 0000100, \quad \beta_6 = 0100000, \quad \beta_7 = 0010000.$$

Set $Z(J_i) = \langle e_i \rangle$ and $J = J_1, \ldots, J_7$. Then $\{e_1, \ldots, e_7\}$ is a set of commuting involutions which span $Z = Z(J)$.

LEMMA 1 (E_7).
(i) $N_G(J)/J \cong L_3(2)$ *and* $N_G(J)$ *is 2-transitive on* Σ, *hence on* $\{e_1, \ldots, e_7\}$.
(ii) *If G is simply connected, the relations on* $\{e_1, \ldots, e_7\}$ *are spanned by* $\{e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6\}$. *So* $|Z| = 2^4$.
(iii) *If G is adjoint, the relations on* $\{e_1, \ldots, e_7\}$ *are spanned by* $\{e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6, e_1e_2e_3\}$. *So* $|Z| = 2^3$.

PROOF. For each i, the centralizer $C_G(J_i)$ is of type D_6. Within D_6 a maximal commuting product of fundamental SL_2's corresponds to a decomposition of the usual orthogonal module into three perpendicular 4-spaces. One checks that S_4 is induced on such a commuting product, transitive on the 6 copies of SL_2. Hence, $N_G(J)$ is 2-transitive on $\{J_1, \ldots, J_7\}$, $N_G(J)/J$ has order 168, and (i) follows.

For (ii) and (iii) first check that $e_4e_5e_6e_7$, $e_2e_3e_6e_7$, $e_1e_2e_5e_6$, $e_1e_2e_3$ are each in $Z(G)$ (show that they centralize each root group corresponding to a fundamental root). Hence, in the simple group $|Z| \le 2^3$. Equality must hold since $L_3(2)$ acts nontrivially on Z. This gives (iii). For (ii), view $E_7 \le E_8$ and note that $e_4e_5e_6e_7$, $e_2e_3e_6e_7$, $e_1e_2e_5e_6$ are in $Z(E_8) = 1$, while $e_1e_2e_3$ is not.

One can now list explicitly all relations on the e_i's, listing tuples of integers to indicate corresponding products of e_i's which are trivial.

(*)

G simply connected: 4567, 2367, 1256, 1247, 2345, 1357, 1346.

G adjoint: 4567, 2367, 1256, 1247, 2345, 1357, 1346,
123, 145, 347, 356, 167, 246, 257,
1234567

LEMMA 2 (E_7). *Let* $E \le J$ *be an elementary abelian 2-group.*
(i) *There exist subgroups* Q_i *of* J_i $(1 \le i \le 7)$ *such that* $Q_i = \langle x_i, y_i \rangle$ *is quaternion of order 8,* $N_{J_i}(Q_i)$ *induces* S_3 *on* Q_i, *and* $E \le Q = Q_1 \ldots Q_7$.
(ii) $|E| \le 2^8, 2^7$ *according to whether G is adjoint or simply connected.*
(iii) *If G is adjoint, there is a unique J-class of elementary abelian groups of order* 2^8, *represented by* $\langle Z, x_4x_5x_6x_7, x_2x_3x_6x_7, x_1x_2x_5x_6, x_1x_2x_3, y_1 \ldots y_7 \rangle$.
(iv) *If G is simply connected, there is a unique J-class of elementary abelian groups of order* 2^7, *represented by* $\langle Z, x_4x_5x_6x_7, x_2x_3x_6x_7, x_1x_2x_5x_6 \rangle$.
(v) *Any 2-group in G is conjugate to a subgroup of* $N_G(J)$.

PROOF. Consider $EZ/Z \leq J/Z$ and project to each of the simple summands. Each projection of E is contained in the Klein 4-subgroup of a group isomorphic to S_4. The preimages of the S_4's are the normalizers of the Q_i's. This gives (i).

For the other parts take E of maximal order. Then $Z \leq E$. Suppose $e \in E - Z$. Conjugating by a suitable element in the product of the normalizers of the Q_i's we may assume e is a product of certain of the elements $x_1, \ldots, x_7$. Since e is an involution the relations force $e = x_i x_j x_k x_l$, $x_i x_j x_k$, or $x_1 \ldots x_7$, where $ijkl$ or ijk is one of the tuples in (*).

For each i, $[x_i, y_i] = e_i$. Moreover, inspection of the above tuples shows: $|\{i, j, k, l\} \cap \{r, s, t\}| = 0$ or 2 and $|\{i, j, k, l\} \cap \{r, s, t, v\}| = 2$ if $\{i, j, k, l\} \neq \neq \{r, s, t, v\}$. The proof of (ii), (iii), and (iv) is completed using these facts and an easy check of cases. Finally, (v) follows since $E \leq N_G(T)$ and the orders of $N_G(T)/T$ and $N_J(T)/T$ have the same 2-part (2^{10}).

LEMMA 3 (E_7). *Assume G is adjoint and $E \leq N_G(J)$ is an elementary abelian 2-group. Then $|E| \leq 2^8$, equality possible only if E is G-conjugate to a subgroup of J.*

PROOF. Suppose $|E| \geq 2^8$, $E \leq N_G(J)$, but $E \not\leq J$. Let $X = EJ/J$, regarded as a subgroup of $L_3(2)$. Hence, $X \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The permutation action of $N_G(J)/J$ on Σ is the same as that on $Z^\#$. Let $Y = E \cap J$, with $Y \leq Q = Q_1 \ldots Q_7$ as in Lemma 2, and E normalizing Q (use the fact that $N_G(J_i) = J_i C_G(J_i)$ for each i). Set $a_i = x_i Z$ and $b_i = y_i Z$.

CASE 1. $C_Z(E) \cong \mathbb{Z}_2$. By transitivity we may assume $C_Z(E) = \langle e_1 \rangle$. Since involutions in $L_3(2)$ have a 2-dimensional fixed space on the usual module, $X \cong Z_2 \times Z_2$. So $Y = E \cap J$ is elementary abelian of order at least 2^6 and $|YZ/Z| \geq 2^5$.

$R = C_{J/Z}(X)$ is the product of groups of type PSL_2, one for each orbit of X on Σ. Now, X has orbits of size $1, 2, 2, 2$. Write $R = R_1 \ldots R_4$, each $R_i \cong PSL_2$ and $R_1 = J_1 Z/Z$. If $\{J_i, J_j\}$ is an orbit, then $e_i e_j$ is fixed by E, hence $e_1 = e_i e_j$. Thus $1ij$ is one of the triples above. So the orbits are $\{J_2, J_3\}$, $\{J_4, J_5\}$, $\{J_6, J_7\}$, with corresponding PSL_2's R_2, R_3, R_4. $YZ/Z \cap R_1 = 1$ (since $Y \cap J_1 = = \langle e_1 \rangle$). So conjugating by an appropriate element of $N(Q)$ we may assume that the image of YZ/Z under projection to $R_2 R_3 R_4$ contains a hyperplane of $\langle a_2 a_3, b_2 b_3, a_4 a_5, b_4 b_5, a_6 a_7, b_6 b_7 \rangle$. Intersecting the projection with $\langle a_2 a_3, b_2 b_3 \rangle$, $\langle a_4 a_5, b_4 b_5 \rangle$, and $\langle a_6 a_7, b_6 b_7 \rangle$, we may assume Y contains elements projecting to $a_2 a_3$, $a_4 a_5$, and $a_6 a_7$. Hence, we may assume Y contains $x_1 x_2 x_3$, $x_1 x_4 x_5$, and $x_1 x_6 x_7$. But also, Y contains an element projecting to an involution in $\langle b_2 b_3, b_4 b_5 \rangle$, forcing Y to be nonabelian. Contradiction.

CASE 2. $C_Z(E) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then E fixes 3 J_i's, but does not centralize Z. So we may assume E normalizes J_1, J_2, and J_3. No element of $L_3(2)$ fixes more than 3 elements of the usual module, so X is semiregular on $\{J_4, J_5, J_6, J_7\}$.

First assume $X \cong \mathbb{Z}_2$. Then YZ/Z has order at least 2^5 and without loss of generality we may assume the nontrivial orbits of E on Σ to be $\{J_4, J_5\}$ and $\{J_6, J_7\}$. Now $Y\cap J_1J_2J_3$ is not contained in Z, so we may assume $x_1x_2x_3 \in Y$. If $Y\cap J_1J_2J_3 = \langle e_1, e_2, x_1x_2x_3\rangle$, then the image of Y under projection to $J_4J_5J_6J_7Z/Z$ coincides with $\langle a_4a_5, b_4b_5, a_6a_7, b_6b_7\rangle$ and this forces Y to be nonabelian. So assume $x_1x_2x_3, y_1y_2y_3$ are both in Y. As above, we may assume Y contains an element projecting to a_4a_5, which again forces Y to be nonabelian. Thus $X\cong \mathbb{Z}_2\times\mathbb{Z}_2$.

Hence E has a unique nontrivial orbit on Σ of size 4 and $|YZ/Z|\geq 2^4$. It follows that $|Y\cap J_1J_2J_3|\geq 2^4$, so we may assume $x_1x_2x_3, y_1y_2y_3\in Y$. Now $N_G(J_1)\cap N_G(J_2)\cap N_G(J_3) = J_1J_2J_3D$, where $D=D^0$ is simply connected of type D_4 (indeed, $Z(D)=\langle e_4e_5, e_5e_7\rangle$). Take $h\in E-\langle Z, x_1x_2x_3, y_1y_2y_3\rangle$. Since h commutes with $x_1x_2x_3$ and $y_1y_2y_3$, we may take $h\in D$. Now D has just 1 class of involutions in $D-Z(D)$, represented by e_4 (corresponding to involutions in SO_8 of type $(1)^4(-1)^4$). Hence $C_D(h)$ is D-conjugate to $C_D(e_4)=J_4J_5J_6J_7$. Thus, $E\leq J_1J_2J_3C_D(h)$, a D-conjugate of J. This completes the proof of Lemma 3.

LEMMA 4 (E_7). *Assume G is simply connected and $E\leq N_G(J)$ is an elementary abelian 2-group. Then $|E|\leq 2^7$, equality possible only if E is G-conjugate to a subgroup of J.*

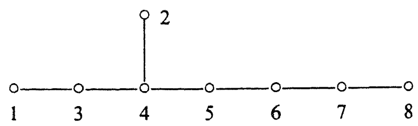
PROOF. Assume $|E|\geq 2^7$ and $E\not\leq J$. Then, up to conjugacy in $N_G(J)/J\cong$ $\cong L_3(2)$, we have that $X=EJ/J$ is one of the groups listed in the table below, where a, b, c are elements of $N_G(J)/J$ inducing the permutations $(2,3)(6,7)$, $(4,5)(6,7)$, $(4,6)(5,7)$, respectively, on Σ. A direct check shows that, in each case, $C_Z(X)$ is as indicated in the table. Thus, the rank of $E\cap Z$ (a subgroup of $C_Z(X)$) is at most 3, 2, and 3, so that $|(E\cap J)Z/Z|\geq 2^3$, 2^3, and 2^2, in the respective cases.

On the other hand, if $q=q_1\dots q_7$, where $q_i\in Q_iZ/Z$, is an involution then the tuple of indices i with $q_i\neq 1$ is a 4-tuple of (*). Moreover, if q is centralized by X, this tuple must be invariant under the permutation action of X on Σ. In the table, under *inv* (X), those tuples from (*) are listed which are X-invariant. It readily follows from the structure of *inv* (X) that $(E\cap Q)Z/Z$ has size at most 2^2 in all three cases. Therefore, we must have $X=\langle b,c\rangle$, $E\geq\langle e_1,e_2,e_3\rangle$, and, without loss of generality, $(E\cap J)Z/Z=\langle x_4x_5x_6x_7, y_4y_5y_6y_7\rangle Z/Z$. In particular $E\leq N_G(J_1)N_G(J_2)N_G(J_3)$, and we can finish as in the previous lemma.

X	$\langle c\rangle$	$\langle a,b\rangle$	$\langle b,c\rangle$
$C_Z(X)$	$\langle e_1,e_2,e_3\rangle$	$\langle e_1,e_2e_3\rangle$	$\langle e_1,e_2,e_3\rangle$
inv (X)	4567, 1357, 1346	4567, 2367, 2345	4567

The E_7 case of the theorem follows from Lemmas 2, 3, and 4.

E_8. We proceed as for E_7. Again T is a maximal torus, and Σ a maximal set of pairwise commuting fundamental SL_2's. We label the diagram



and take $\Sigma=\{J_1,\dots,J_8\}$, where $(J_i)_{1\le i\le 7}$ as for E_7 and $J_8=\langle U_{\pm\beta_8}\rangle$, with $\beta_8=23465432$. Set $Z(J_i)=\langle e_i\rangle$ and $J=J_1\dots J_8$. Then $\{e_1,\dots,e_8\}$ is a set of commuting involutions spanning $Z=Z(J)$.

LEMMA 5 (E_8).

(i) *$N_G(J)/J\cong\mathbb{Z}_2^3L_3(2)$ and $N_G(J)$ is 3-transitive on Σ, hence on $\{e_1,\dots,e_8\}$.*

(ii) *The relations on $\{e_1,\dots,e_8\}$ are given by the tuples of even length in (*) and the tuples obtained by joining 8 to the tuples of odd length in (*).*

PROOF. For each $i\in\{1,\dots,8\}$, the group $C_G(J_i)$ is of type E_7, so the lemma is easily derived from Lemma 1.

LEMMA 6 (E_8). *Let $E\le J$ be an elementary abelian 2-group.*

(i) *There exist subgroups Q_i of J_i such that $Q_i=\langle x_i,y_i\rangle$ is quaternion of order 8, $N_{J_i}(Q_i)$ induces S_3 on Q_i, and $E\le Q=Q_1\dots Q_8$.*

(ii) *$|E|\le 2^9$.*

(iii) *There is a unique J-class of elementary abelian subgroups of order 2^9, represented by $\langle Z,x_4x_5x_6x_7,x_2x_3x_6x_7,x_1x_2x_5x_6,x_1x_2x_3x_8,y_1\dots y_8\rangle$.*

(iv) *Any 2-group in G is conjugate to a subgroup of $N_G(J)$.*

PROOF. Similar to Lemma 2.

LEMMA 7 (E_8).

(i) *Let $K=\langle e_je_k|1\le j,k\le 8\rangle$ and $R/J=O_2(N(J)/J)$. Then K is a hyperplane in Z and $R=N(J)\cap C(K)$.*

(ii) *$R-J$ contains a conjugate d of e_1 such that each involution in $R-J$ is $N(J)$-conjugate to an involution in dK.*

(iii) *If $ijkl$ is a 4-tuple as in Lemma 5(ii) and if x_i,x_j,x_k,x_l are elements of order 4 in J_i,J_j,J_k,J_l, respectively, then $x_ix_jx_kx_l\in e_1^G$.*

PROOF. $N(J)$ acts on K since it permutes Σ, and clearly K is a hyperplane in Z. So $N_G(J)$ induces $L_3(2)$ on K and (i) follows. Observe that R/J acts regularly on Σ.

Let $z\in K^{\#}$. Then $J\le D=C_G(z)=D_8$ (half-spin). Consider SO_{16} (an image of the covering group of D) and its subgroup $\tilde D=SO_{16}\cap(O_4)^4$. Set $(\tilde D)^0=\tilde J$, a group corresponding to J. Choose reflections t_1,t_2,t_3,t_4, one from each O_4. The product of any two of these is in SO_{16}, and these products generate an elementary abelian group $\tilde S$ of order 8 which acts faithfully on the set $\tilde\Sigma$ of simple factors of $\tilde J$. Let $\tilde R$ denote the subgroup corresponding to R. Then $\tilde S\cap\tilde R$ is not contained in $\tilde J$. Since $t=t_1t_2t_3t_4$ is the unique element in $\tilde S$ acting semiregularly on $\tilde\Sigma$, we have $t\in\tilde R-\tilde J$. In SO_{16}, t is conjugate to an involution in a fundamental SL_2. Translating this to D we conclude that there must exist an element $d \in (D \cap R) - J$, with d a conjugate of e_1.

To prove (ii) let t be any involution in $R - J$. Since $(R/J)^{\#}$ is fused in $N_G(J)$, we may assume $tJ = dJ$. Hence, J/Z is the direct product of simple groups permuted semiregularly by t. Therefore, all involutions in dJ are conjugates of those in dZ. Hence, we may assume $t \in dZ$. Also, $C_Z(d) = K$ (since $Z - K = \{e_1, \dots, e_8\}$). So the only involutions in dZ are in fact in dK. This proves (ii).

For (iii) again consider D and choose $X \circ Y \leq D$ with X, Y of type D_4. Then X and Y are simply connected and we may take $J \leq X \circ Y$, where $J \cap X$ and $J \cap Y$ are each a product of 4 of the fundamental SL_2's. Say $J \cap X = J_r J_s J_u J_v$. One checks that $e_r e_s e_u e_v = 1$ so $rsuv$ is one of the 4-tuples of Lemma 5(ii). From 3-transitivity of $N(J)$ on Σ we may assume $\{r, s, u, v\} = \{i, j, k, l\}$. Set $x = x_i x_j x_k x_l$. The image of x in a quotient of X isomorphic to SO_8 is necessarily conjugate to the images of e_i, e_j, e_k, and e_l (by consideration of the action of this image on the orthogonal module). Without loss we may assume the kernel to the map is $\langle e_i e_j \rangle$. Hence, $x \sim e_i$ or $e_i(e_i e_j) = e_j$, proving (iii).

LEMMA 8 (E_8). *Let $E \leq N_G(J)$ be an elementary abelian 2-group. Then $|E| \leq 2^9$, equality possible only if E is G-conjugate to a subgroup of J.*

PROOF. Assume $|E| \geq 2^9$ and let $X = EJ/J$. If X has a fixed point, say J_8, on Σ, then $E \leq N(J_8) = J_8 E_7$ and we are done by reduction to E_7. Similarly, we may assume E centralizes no conjugate of e_1.

Assume $X \cap (R/J) = 1$, so $|X| \leq 4$. Involutions in $N(J)/J$ fixing a point in Σ fix exactly 4 points, so from the above paragraph we conclude X contains a regular involution, say x. Then $C_Z(x) \leq K$ (as $Z - K = \{e_1, \dots, e_8\}$) and x is nontrivial on K (as $x \notin R$). Thus, $|E \cap Z| \leq |C_Z(E)| \leq 4$. But $|E \cap J| \geq 2^7$, whence $(E \cap J)Z$ is an elementary abelian group of order at least 2^9.

Apply Lemma 6. Replacing E by a J-conjugate, if necessary, we may assume $(E \cap J)Z/Z = \langle Z, x_4x_5x_6x_7, x_2x_3x_6x_7, x_1x_2x_5x_6, x_1x_2x_3x_8, y_1 \dots y_8 \rangle$. However, x must centralize $(E \cap J)Z/Z$ and have no fixed points on Σ. Checking possible orbits of x we see this to be impossible.

We may now assume $X \cap (R/J) \neq 1$ and let $s \in (E \cap R) - J$. Lemma 7(ii) implies $sK = aK$ for some $a \in e_1^G$. From the first paragraph it follows that E does not centralize K. In particular, E is not contained in R. Let $f \in E - R$. Then $|E \cap Z| \leq |C_Z(E)| \leq |C_Z(s) \cap C_Z(f)| = |C_K(f)| = 4$.

It follows that $E \cap J$ must contain an element of the form $d = x_i x_j x_k x_l z$, where $ijkl$ is a tuple as in Lemma 5, x_r is of order 4 in J_r for $r \in \{i, j, k, l\}$, and $z \in Z$. Note that $\{i, j, k, l\}$ is necessarily a union of two orbits of $\langle s \rangle$. Also X must act on $\{i, j, k, l\}$ and also on its complement (as E centralizes s).

If $|X| \leq 4$, then as above $|(E \cap J)Z| \geq 2^9$ and we again obtain (recall that $|E \cap Z| \leq 4$) a contradiction using Lemma 6. Hence, $|X| \geq 8$. Restricting the abelian group X to $\{i, j, k, l\}$ we obtain an element $1 \neq x \in X$ fixing i, j, k, and l. Also X is transitive on either $\{i, j, k, l\}$ or its complement $\{i', j', k', l'\}$.

Order considerations imply $E\cap J$ must also contain an element of the form $x_{i'}x_{j'}x_{k'}x_{l'}z'$, so we may assume X is transitive on $\{i,j,k,l\}$. Let $e\in E$ satisfy $eJ=x$. Then $d=d^e=(x_ix_jx_kx_l)^e z^e$. But e normalizes each of J_i, J_j, J_k, J_l, so $x_i^e=x_ie_i^t$ for $t=0,1$. Transitivity forces $x_r^e=x_re_r^t$ for each $r\in\{i,j,k,l\}$, and so $(x_ix_jx_kx_l)^e=x_ix_jx_kx_l$. Thus, $z=z^e$ and so $z\in C_Z(e)=\langle e_i,e_j,e_k,e_l\rangle$. Hence $d\sim x_ix_jx_kx_l$ (use an element of $\langle y_i,y_j,y_k,y_l\rangle$) and so by Lemma 7(iii), $d\in e_1^G$, contradicting the first paragraph.

The E_8 case of the theorem is now immediate from Lemmas 6 and 8.

COROLLARY. *Let q be an odd prime power. Then the 2-rank of ${}^2G_2(q)$, $G_2(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $\hat{E}_7(q)$, $E_8(q)$ is $3,3,5,6,6,8,7,9$ in the respective cases.*

PROOF. Let G be the algebraic group and let q be a power of the prime p. If σ is a field endomorphism, then it is immediate from the description given that the elementary abelian 2-groups of maximal rank can be taken in $O^{p'}(G_\sigma)$. Suppose G is of type E_6 and that $\sigma=q\tau$, where τ is a graph automorphism. Set $E=\Omega_2(T)$, where T is a σ-stable torus contained in a σ-stable Borel subgroup. Let $\dot{w}_0\in N(T)$ represent the long word $w_0\in W=N(T)/T$. Since $\tau\dot{w}_0$ acts on T by inversion it fixes E elementwise; hence $\sigma\dot{w}_0=q\tau\dot{w}_0$ fixes E elementwise. The result follows since Lang's Theorem implies that σ and $\sigma\dot{w}_0$ are G-conjugate. Finally, consideration of the centralizer of an involution shows that the 2-rank of ${}^2G_2(q)$ is 3.

2. ODD PRIMES

In this section r is an odd prime and G is an algebraic group of exceptional type over an algebraically closed field of characteristic distinct from r. We begin with a general lemma.

LEMMA 9. *Let J be an algebraic group over an algebraically closed field of characteristic $p\neq r$. Suppose $J=T_s\circ J_1\circ\cdots\circ J_k$, a central product of an s-dimensional torus and k groups isomorphic to SL_r. Then the r-rank of J is $s+k(r-1)$ and all elementary abelian r-subgroups of maximal rank are contained in a maximal torus of J.*

PROOF. Assume $2<r\neq p$. J contains a maximal torus of rank $s+k(r-1)$, so the r-rank of J is at least $s+k(r-1)$. Since the r-rank of both SL_r and PSL_r is easily checked to be $r-1$, the first assertion follows by induction, factoring out T_s and all but one of the SL_r's. These remarks also show that J/J_i has r-rank $s+(k-1)(r-1)$, for each $1\leq i\leq k$. Let E be an elementary abelian r-subgroup of J having maximal rank and fix i $(1\leq i\leq k)$. By the above, $E\cap J_i$ has rank $r-1$, and since $E\cap J_i$ is abelian it is contained in a maximal torus T_i' of $J_i\cong SL_r$. Moreover, $C_{J_i}(E\cap J_i)=T_i'$. Hence, $E\leq\bigcap_i C_J(E\cap J_i)=T_sT_1',\dots,T_k'$, a maximal torus of J. The lemma follows.

By the results of [Springer & Steinberg, 1970] every elementary abelian r-group in G can be embedded in a torus if $r>3$ for G_2 and F_4, $r>5$ for E_6 and E_7, and $r>7$ for E_8. Thus, we only consider the remaining odd primes. Let T be a maximal torus of G. The following subgroups D of G contain T and are such that $N_D(T)/T$ contains a Sylow r-group of $N_G(T)/T$.

G	$r=3$	$r=5$	$r=7$
G_2	A_2		
F_4	A_2A_2		
E_6	$(A_2A_2A_2)3$	T_2A_4	
E_7	$T_1(A_2A_2A_2)3$	T_3A_4	
E_8	$A_2(A_2A_2A_2)3$	A_4A_4	T_2A_6

Here, T_i stands for a torus of rank i, and A_{r-1} for a fundamental subgroup isomorphic to SL_r.

PROPOSITION. *If r is an odd prime and G is an algebraic group of exceptional Lie type over an algebraically closed field of characteristic $\neq r$, then the r-rank of G is the Lie rank of G. Moreover, all elementary abelian r-subgroups of maximal rank are conjugate and contained in a maximal torus of G.*

PROOF. Let E be an elementary abelian r-subgroup of G of rank at least the Lie rank of G. In view of the previous comments we may take r to be one of the primes in the table above and assume that $E\leq D$, where D is also given in the table. A dimension check shows that D^0 contains a maximal torus of G, so the result follows from Lemma 9 provided $E\leq D^0$.

Suppose there is $e\in E-D^0$. Then $r=3$, $G=E_6$, E_7, or E_8. Here D contains a normal subgroup S with $S\cong 1$, T_1, or A_2, respectively, $D/SZ(D)$ the wreath product of PSL_3 with $\mathbb{Z}_3$, and $C_{D/SZ(D)}(e)\cong PSL_3\times\mathbb{Z}_3$. So the 3-rank of E is at most 3 plus the 3-rank of $C_{SZ(D)}(e)$. From the action of e it is clear that the latter is at most 2, 3, 4, respectively, so this is a contradiction.

REFERENCES

Aschbacher, M. and G.M. Seitz – Involutions in Chevalley groups over fields of even order, Nagoya J. Math. **63**, 1-91 (1976).

Borel, A. – Oeuvres, Collected papers, I, 1948-1958, Springer, Berlin (1983).

Borel, A. and J.-P. Serre – Sur certains sous-groupes des groupes de Lie compacts, Comment. Math-Helv. **27**, 128-139 (1953).

Springer, T.A. and R. Steinberg – Conjugacy Classes, Part E in: Seminar on Algebraic Groups, and Related Finite Groups (A. Borel et al.) Springer Lecture Notes in Math. **131**, Springer, Berlin (1970).